Noncyclic Covers of Knot Complements

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Abstract

Hempel has shown that the fundamental groups of knot complements are residually finite. This implies that every nontrivial knot must have a finite-sheeted, noncyclic cover. We give an explicit bound, $\Phi(c)$, such that if K is a nontrivial knot in the three-sphere with a diagram with c crossings and a complement with a particularly simple JSJ decomposition then the complement of K has a finite-sheeted, noncyclic cover with at most $\Phi(c)$ sheets.

1 Introduction

Let K be a nontrivial knot in S^3. Let $M = S^3 - N(K)$ be the complement of an open regular neighborhood of K. It is well known that for each positive integer, k, M has a unique cyclic cover with k sheets arising from the map $\pi_1(M) \to H_1(M) \cong \mathbb{Z} \to \mathbb{Z}/k\mathbb{Z}$. However, much less is known about the noncyclic covers of knot complements. In [7] Hempel establishes that fundamental groups of Haken 3-manifolds are residually finite. This shows in particular that the fundamental groups of knot complements are residually finite. Thus for any nontrivial element, g, of the commutator of $\pi_1(M)$ there must be a nontrivial normal subgroup of finite index in $\pi_1(M)$ not containing g. It follows that knot complements must have infinitely many finite, nonabelian covers. The goal of this exposition is to give an explicit function $\Phi(c)$ such that if K is a nontrivial knot with a diagram with c crossings and its complement $M = S^3 - N(K)$ has a simple JSJ decomposition then M has a noncyclic cover with at most $\Phi(c)$ sheets.

The question behind this investigation is how well finite index subgroups differentiate the fundamental group of a nontrivial knot complement from the group of integers. The bound on the index given here (see Theorem 1) seems to be largely an artifact of the techniques used. It is safe to conjecture that it might be drastically improved with another approach.

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If one could generalize these results to all knots then in a technical sense, one would get an algorithm for detecting knottedness as follows: If one starts with a knot with c crossings and systematically creates all covers of the complement with $\Phi(c)$ or less sheets then if a noncyclic cover is found, the knot is nontrivial. If no such cover is found, the knot is trivial. However, in light of the large bound given in this paper, much better, if still impractical algorithms already exist to establish knottedness. In [6], a bound on the number of Reidemeister moves needed to convert an arbitrary diagram of the unknot to the standard diagram of the unknot is given. Also, Ian Agol has shown that computing lower bounds for the genus of a knot is in NP. At this writing, it is an open problem whether there is a practical algorithm to detect knottedness.

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2 Main Result

Set

$$A(n) = \frac{(n^2-n+1)!}{n^2[(n-1)!]^n}, \tag{1}$$

$$B(n) = \frac{n^3-n^2}{n^2-n+1}, \tag{2}$$

$$\begin{aligned} D(n) &= \\ &\exp\Big[2(4n+4)\left(8n^2+4n\right)^{2^{4n+4}} A(4n+5)(27n+5) \\ &\cdot\left(2^{4n+2}+3\cdot 2^{3n+3}+\left(\tfrac{\sqrt{3}}{2}+3\right)n+B(4n+5)\log 2+\tfrac{\sqrt{3}}{2}\right) \\ &+2^{4n+4}\left(8n^2+4n\right)^{(4n+4)2^{4n+4}}\Big(2\log 2+4\log(16n^2 3^{n-1}) \\ &+3(2^{4n-1}-1)(\log 2)+3(2^{4n}+2^{4n-1}-2)A(4n+5)(27n+5) \\ &\cdot\left(2^{4n+2}+3\cdot 2^{3n+3}+\left(\tfrac{\sqrt{3}}{2}+3\right)n+B(4n+5)\log 2+\tfrac{\sqrt{3}}{2}\right)\Big)\Big] \end{aligned} \tag{3}$$

and

$$\Phi(c) = \left(87\left(\log(D(100c))+8c\log\tfrac{c}{2}\right)\right)^{24c\left(2^{4n+4}\left(8n^2+4n\right)^{(4n+4)2^{4n+4}}\right)}. \tag{4}$$

A knot K in S^3 will be called *decompositionally linear* if the JSJ decomposition of its complement along essential tori $T_1, T_2, \cdots, T_r$ has the property that in S^3, K and T_i are on the same side of T_j if $j < i$.

The main result is as follows:

Theorem 1. *Let K be a nontrivial, decompositionally linear knot in S^3 and $M = S^3 - K$ its complement. Suppose K has a diagram with c crossings. Then M has a noncyclic cover with at most $\Phi(c)$ sheets.*

The proof of Theorem 1 will proceed using Thurston's geometrization for knot complements. Let K and M be as in the statement of the theorem. Then the JSJ decomposition of M cuts M along essential tori $T_1, T_2, \cdots, T_r$ into spaces $M_0, M_1, \cdots, M_r$ where either M_i is Seifert fibered or $M_i - \partial M_i$ has a complete hyperbolic structure.

3 Topology of knot complements

3.1 Standard spines and ideal triangulations

For our purposes an *ideal triangulation* of a 3-manifold, M, will be a simplicial complex, $\mathcal{T}$, satisfying some further conditions. The complex, $\mathcal{T}$, must be a union of a finite number of 3-simplices with pairs of faces identified. In fact, we insist that there are no unidentified "free" faces. Identification of different faces of the same tetrahedron will be allowed. For $\mathcal{T}$ to be an ideal triangulation of the 3-manifold, M, we require $\mathcal{T}$ minus its vertices to be homeomorphic to $M - \partial M$. We will write $\mathcal{T} = \bigcup_{i=1}^{n} \sigma_i$ to indicate that $\mathcal{T}$ is an ideal triangulation with the n ideal tetrahedra, $\sigma_1, \sigma_2, \ldots, \sigma_n$. For our purposes, the links of the vertices in our ideal triangulations will always be tori, and all 3-manifolds and their ideal triangulations will be orientable.

When dealing with the geometric pieces of M it will be convenient to have a bound on the number of tetrahedra needed to triangulate them. In working with ideal triangulations it is often helpful to be familiar with the dual notion of standard spines. As in [2] a *spine* is simply a 2-complex. The *singular 1-skeleton* of a spine is the set of points which do not have neighborhoods homeomorphic to open disks. The *singular vertices* of a spine are the points of the singular 1-skeleton which do not have neighborhoods in the singular 1-skeleton homeomorphic to open intervals.

Let C be a spine, C_1 be the singular 1-skeleton of C, and C_0 be the singular vertices of C. The spine C will be a *standard spine* if it satisfies three conditions. Firstly, C must satisfy the *neighborhood condition*. That is, every point of C must have a neighborhood homeomorphic to one of the three 2-complexes pictured in Figure 1. Secondly, $C - C_1$ must be a union of countably many disjoint 2-disks. Thirdly, we require that $C_1 - C_0$ be a union of countably many disjoint arcs. The complex, C, is a *spine* (resp. *standard spine*) *of a 3-manifold* N if $C \subset N$ is a spine (resp. standard spine) and N collapses to C. An important property of a standard spines is that if C is a standard spine of N

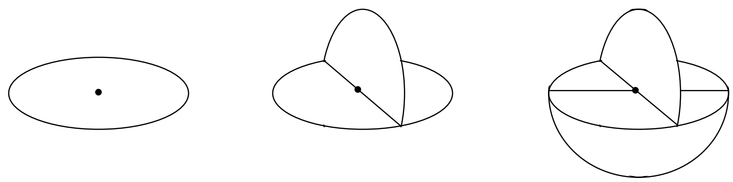

Figure 1: The three possible neighborhoods in a standard spine

then if C is embedded in any 3-manifold then N is homeomorphic to a regular neighborhood of C in that manifold.

In [19] and [14] it is mentioned that standard spines are dual to ideal triangulations (see Figure 2). For every ideal triangulation of a 3-manifold there

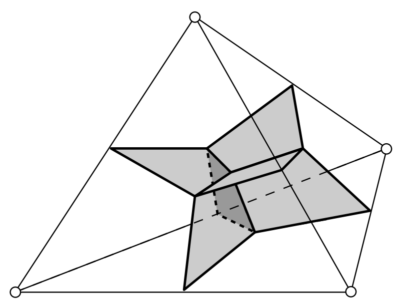

Figure 2: An ideal tetrahedron and its dual spine

is a dual standard spine, and for every standard spine there is a dual ideal triangulation. Thus we see that a standard spine carries the same information as an ideal triangulation. Moreover, the number of singular vertices in a standard spine will be the number of ideal tetrahedra in the dual triangulation. We will exploit this duality a number of times.

3.2 Triangulating a knot complement

A preliminary step in our exposition will be to relate the number of crossings in a knot diagram to the number of ideal tetrahedra needed to triangulate the complement of the knot. It is noted in [14] that the number of ideal tetrahedra needed is at most linear in the number of crossings in a projection. Here we give an explicit relationship. The argument is essentially based on the triangulation algorithm in Jeff Weeks' program, SnapPea.

Lemma 1. *Let K be a knot in S^3 with a diagram with $c > 0$ crossings. Then the complement of K has an ideal triangulation with less than $4c$ ideal tetrahedra.*

A proof of this lemma is given in Appendix A.1.

Now we translate the bound on the number of ideal tetrahedra needed to triangulate M into a bound on the number needed to triangulate the geometric pieces of M.

Lemma 2. *Let K be a knot in S^3 and $M = S^3 - K$ its complement. Suppose M can be triangulated with t ideal tetrahedra. Also suppose that embedded, disjoint tori, $T_1, T_2, \cdots, T_r$, give the JSJ decomposition of M, and $M_0, M_1, \cdots, M_r$ are the connected components after cutting. Then the M_i's have ideal triangulations with t_i ideal tetrahedra each so that $\sum_{i=0}^{r} t_i \leq 25t$.*

Proof. Let K and M be as in the statement of the lemma. By assumption M has an ideal triangulation $\mathcal{T} = \bigcup_{i=1}^{t} \sigma_i$ with t ideal tetrahedra. We may choose our tori $T_1, T_2, \cdots, T_r$ so that their union S is a normal surface with respect to $\mathcal{T}$. For each $i, 1 \leq i \leq t$, S cuts σ_i into pieces with four basic types (see Figure 3):

(a) Pieces whose closure intersects S in two triangles.

(b) Pieces whose closure intersects S in two quadrilaterals.

(c) Pieces whose closure intersects S in two triangles and one quadrilateral.

(d) Pieces whose closure intersects S in four triangles.

Of course there will also be pieces whose closures in σ_i will be incident with the corners of σ_i, but we will put these pieces in categories **(a)** - **(d)** based on how they look when the corners of σ_i are cut off by triangles.

We will now construct a standard spine of $M - S$. Consider the ith tetrahedron in the ideal triangulation of M and its intersection with the surface S. For each region of type **(a)** and **(b)** place a triangular or quadrilateral disk in its center parallel to the faces incident with S as shown in Figure 3. For each region of type **(c)** and **(d)** place a 2-complex as in Figure 3. Let $D_i \subset \sigma_i$ be the union of all of these spine pieces. One sees immediately that $C' = \bigcup_{i=1}^{t} D_i$ is a spine of $M - S$ and that it satisfies the neighborhood condition. The spine, C', has a singular vertex for each region of type **(d)**. Clearly each tetrahedron of $\mathcal{T}$ contains at most one region of type **(d)**. Thus C' has at most t vertices. Note that each connected component of C' has a nonempty singular 1-skeleton, for if there were a component, A, of C' with empty singular 1-skeleton then the component of $M - S$ containing A would be homeomorphic to the torus crossed with the open unit interval.

Let C'_1 be the singular 1-skeleton of C'. Although C' satisfies the neighborhood condition, C' will not in general be a standard spine since $C' - C'_1$ may not be a union of disks. A component of $C' - C'_1$ must have genus 1 or 0 since the boundary components of $M - S$ are tori. Thus if a component of $C' - C'_1$ has b boundary components it may be cut into disks with $b + 1$ or less arcs. For each of these arcs, γ, modify C' as in Figure 4 (This is possible since each connected component of C' has a nonempty singular 1-skeleton). Let C be the modified spine. The modification in Figure 4 takes one spine to another [15] so

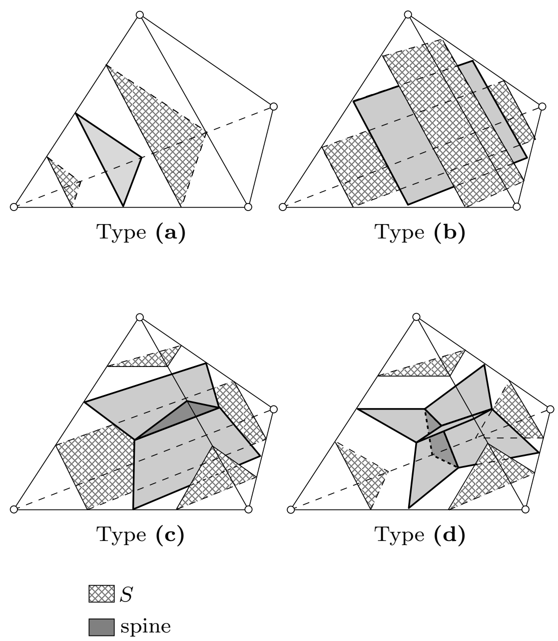

Figure 3: Regions of type **(a)** - **(d)** and their spines

C is a spine of $M - S$. The spine, C, satisfies the neighborhood condition. Let C_1 be the singular 1-skeleton of C and C_0 be the singular vertices of C_1. By construction, $C - C_1$ is a disjoint union of disks. We claim that $C_1 - C_0$ must be a disjoint union of arcs. If not then $C_1 - C_0$ contains an S^1. Since $C - C_1$ is composed entirely of disks, each disk in the component of C containing this S^1 must have this S^1 as its boundary. This is impossible because, as mentioned before, the only such spine satisfying the neighborhood condition is three disks glued along their boundary. This is not a spine of the complement of a knot in the 3-sphere or the complement of a knot in a solid torus. Consequently, C is a standard spine of $M - S$.

Let us now count the singular vertices of C. The total number of boundary components of $C' - C_1'$ is less than or equal to 3 times the number of regions of type **(c)** plus 6 times the number of regions of type **(d)**. The number of components of $C' - C_1'$ is bounded by that same number. Let $\mathcal{A}$ be the set of connected components of $C' - C_1'$, and for $A \in \mathcal{A}$ let b_A be the number of boundary components of A. If a is the number of arcs needed to cut $C' - C_1'$

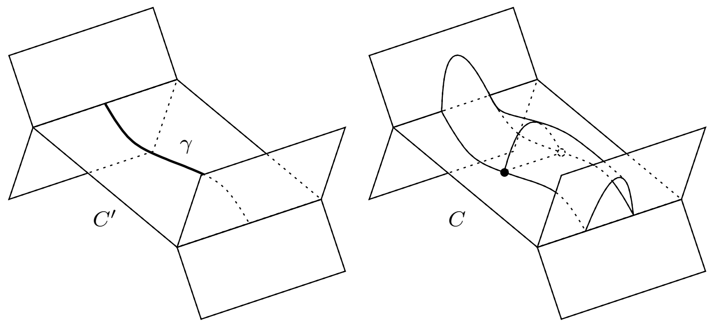

Figure 4: Modifying C' to get C

into disks we have:

$$\begin{aligned}
a &= \sum_{\substack{A\in\mathcal{A}\\ \mathrm{g}(A)=1}} (b_A+1) + \sum_{\substack{A\in\mathcal{A}\\ \mathrm{g}(A)=0}} (b_A-1) \\
&= \sum_{A\in\mathcal{A}} b_A + \sum_{\substack{A\in\mathcal{A}\\ \mathrm{g}(A)=1}} 1 - \sum_{\substack{A\in\mathcal{A}\\ \mathrm{g}(A)=0}} 1 \\
&\leq 6t + |\mathcal{A}| \\
&\leq 12t
\end{aligned}$$

Changing C' to C introduces 2 singular vertices for each cutting arc so C has at most $2a \leq 2\cdot 12t = 24t$ more singular vertices than C'. As mentioned above, C' has at most t vertices so C has at most $25t$ vertices.

The standard spine, C, is dual to an ideal triangulation of $M - S$ with the same number of ideal tetrahedra as singular vertices of C. This shows that $M - S$ can be triangulated with $25t$ or less tetrahedra. □

In proof of Lemma 2 we saw that for each i, $M_i \cap C'$ must have nonempty singular 1-skeleton. The singular 1-skeleton of C' can have, at most, 2 components for every tetrahedron in $\mathcal{T}$ so we get the following corollary:

Corollary 1. *Let K be a knot in S^3 and $M = S^3 - K$ its complement. Suppose M can be triangulated with t ideal tetrahedra. Also suppose that the embedded, disjoint tori $T_1, T_2, \cdots, T_r$ give the JSJ decomposition of M. Then $r < 2t$.*

3.3 The JSJ decomposition of certain knot complements

In general the JSJ decomposition of a knot complement can be quite complicated. For this study we will restrict to a class of knots whose complements have a particularly nice JSJ decomposition. As before, let K be a knot and $M = S^3 - K$ its complement. Let $T_1, T_2, \cdots, T_r \subset M \subset S^3$ be the tori in the decomposition for M. We restrict K (and possibly re-index the tori) so that K

and T_i are on the same side of T_j whenever $j < i$. These knots will be called *decompositionally linear.* Thus we get M as a graph product of CW complexes based on the graph in Figure 5 (See [7] for more on graph products).

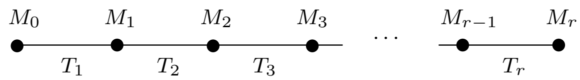

Figure 5: M as a graph product

In order to create a covering space of M, we will produce a compatible collection $\{\widetilde{M_i}\}$ of finite covers of each of the M_i's and assemble them into a finite cover, $\widetilde{M}$, of M following [7] (see Figures 6 and 7). More specifically, we will choose a prime, p, and let $P \trianglelefteq \mathbb{Z} \times \mathbb{Z}$ be the characteristic subgroup generated by $(p, 0)$ and $(0, p)$. For each torus, T_k, we will let $\widetilde{T_k}$ be the cover associated to the subgroup of $\pi_1(T_k) \cong \mathbb{Z} \times \mathbb{Z}$ corresponding to P. We will then produce a finite cover, $\widetilde{M_i}$, of each M_i all of whose boundary components will be equivalent to the appropriate $\widetilde{T_k}$. Finally we will assemble copies of these $\widetilde{M_i}$'s to get a cover of M. For each M_i the challenges will be to discover for which primes, p, we will be able to produce such a cover and then to bound the number of sheets in that cover.

At this point we fix some notation which we use for the rest of the discussion. As mentioned above M_i has one or two boundary components which we denote $\partial_0 M_i = T_{i+1}$ and possibly $\partial_1 M_i = T_i$. Recall that M_i is a subset of S^3. For $k \in \{0, 1\}$ let $E_k(M_i)$ be the closure of the component of $S^3 - \partial_k M_i$ disjoint from M_i. Note that $E_0(M_i)$ is a solid torus, and $E_1(M_i)$ is a knot complement if it is defined. A *meridian*, m_k, of M_i is an essential, simple, closed curve in $\partial_k M_i$ with m_k homologically trivial in $E_k(M_i)$. A *longitude*, l_k, is an essential, simple, closed curve in $\partial_k M_i$ intersecting a meridian once with the added property that a parallel copy of l_k in the interior of M_i has linking number 0 with l_k in S^3. When convenient, we will assume that m_k and l_k are oriented loops.

4 Covers from homology

Let $N = M_i$ be some piece in the JSJ decomposition of the knot complement, M. In many cases we get an appropriate finite sheeted covering space $\widetilde{N}$ for N from homology. The following lemma addresses these cases.

Lemma 3. *Suppose N is the complement of an open, regular neighborhood of a knot L in a solid torus. Suppose further that L has winding number $w \neq 0$ in the solid torus. Then for every prime p not dividing w, N has a p^2-sheeted covering space $\widetilde{N}$, such that each boundary component of $\widetilde{N}$ is the cover of a boundary component of N corresponding to the subgroup $\langle (p, 0), (0, p) \rangle < \mathbb{Z} \times \mathbb{Z} \cong \pi_1(\partial N)$.*

Proof. As in the statement of the lemma, let N be the complement of an open, regular neighborhood of a knot L in a closed solid torus V. Suppose that L

has winding number $w \neq 0$ in the solid torus. Fix an embedding of V in S^3 and define $\partial_0 N$ and $\partial_1 N$ as in the previous section. Let m_0, m_1, l_0, and l_1 be meridians and longitudes of N, and denote their classes in $H_1(N)$ by $[m_0]$, $[m_1]$, $[l_0]$, and $[l_1]$. As shown in Appendix A.2, $H_1(N)$ has abelian presentation

$$\begin{aligned} H_1(N) &= \left\langle [m_0],[m_1],[l_0],[l_1] \middle| [l_1] = w\cdot[m_0], [l_0] = w\cdot[m_1] \right\rangle \\ &= \langle [m_0],[m_1] \rangle. \end{aligned}$$

Let p be a prime not dividing w. Set

$$\bar{\theta} : \pi_1(N) \to H_1(N)/pH_1(N)$$

to be the composition of the Hurewicz homomorphism and the quotient map. The following diagram commutes:

$$\begin{tikzcd} H_1(\partial_k N) \cong \pi_1(\partial_k N) \arrow[r, "i_k^*"] \arrow[d, "i_{k*}"'] & \pi_1(N) \arrow[d, "\bar{\theta}"] \\ H_1(N) \arrow[r, "\mathrm{mod}\, p"'] & H_1(N)/pH_1(N) \end{tikzcd}$$

i_{k*} is an injection with image $U = \langle [m_k],[l_k] \rangle = \langle [m_k], w[m_{1-k}] \rangle$, and $V = \ker(\mathrm{mod} p)$ is generated by $p[m_k]$ and $p[m_{1-k}]$. Note that $V \cap U = \langle [m_k], pw[m_{1-k}] \rangle$. This implies that $\ker((\mathrm{mod}\, p) \circ i_k^*) = \ker(\bar{\theta} \circ i_k^*)$ is the characteristic subgroup of index p^2 in $\pi_1(\partial_k N)$. It follows that the boundary components of the cover $\widetilde{N}$ of N corresponding to $\ker(\bar{\theta}) < \pi_1(N)$ are as prescribed in the statement of the lemma. Also, $\widetilde{N}$ has $|H_1(N)/pH_1(N)| = p^2$ sheets. □

We now consider which primes cannot divide the nonzero winding number of a piece in the satellite (JSJ) decomposition of the complement of K. A bound on the winding number in the lemma above will be of use.

Lemma 4. *Suppose N is a piece in the JSJ decomposition of the complement of a nontrivial knot in S^3 with a c-crossing diagram. Suppose further that N is the complement of an open regular neighborhood of a knot in a solid torus with winding number w in the solid torus. Then $w \leq \frac{c}{2}$.*

Proof. Let K be a nontrivial knot in S^3 and N be a piece of the JSJ decomposition of $M = S^3 - N(K)$. Further, suppose that N has two boundary components. Let M' be the component of $M - \partial_0 N$ which is disjoint from ∂M. Then M' is the complement of a knot, K', in S^3, and K is a satellite of K'. By Theorem 3 of [18] the bridge number, b', of K' must be less than or equal to the bridge number, b, of K.

Consider the knot K'. It has a further satellite knot decomposition since N has two boundary components. Let K'' be the companion for this decomposition, and let b'' be its bridge number. The winding number of K' in the solid torus is w. Using [18, Theorem 3] we can conclude that

$$wb'' \leq b'.$$

Clearly $2 \leq b''$ so

$$2w \leq wb'' \leq b' \leq b.$$

The bridge number of a knot must be less than or equal to the crossing number of any projection of the knot so $b \leq c$; hence, we get the desired result:

$$2w \leq c$$

$\square$

The covers given by Lemma 3 are quite nice in that they can be made to have few sheets relative to the crossing number of our original knot. These are the easy cases because homology does all the work for us. Let us consider the types of pieces of the JSJ decomposition of the knot complement, M, which are not covered by Lemma 3. Given our restriction on K, there are three remaining cases:

1. M_i is a hyperbolic knot complement.
2. M_i is hyperbolic and the complement of a knot in a solid torus with winding number 0 in the solid torus.
3. M_i is a torus knot complement.

We will see in section 6.1 that any Seifert fibered piece of M which is the complement of a knot in a solid torus satisfies the hypotheses of Lemma 3.

5 Hyperbolic Pieces

5.1 Mahler measure and height

In order to address the case in which M_i is a hyperbolic manifold we will use the number of tetrahedra in an ideal triangulation of M_i to limit certain quantities related to a representation of $\pi_1(M_i)$ into $SL_2(\mathbb{C})$. In the process we will encounter certain polynomial equations and algebraic numbers. For numerous reasons the most natural notions of complexity for polynomials and algebraic numbers are given by the Mahler measure and height, respectively. We define these notions here.

Let $P = P(X_1, X_2, \cdots, X_n)$ be a polynomial with complex coefficients. As in [11] the *Mahler measure*, $M(P)$, is given by

$$M(P) = \exp\left(\int_0^1 \cdots \int_0^1 \log|P(e^{2\pi i t_1}, \cdots, e^{2\pi i t_n})|\, dt_1 \cdots dt_n\right).$$

As mentioned above, the Mahler measure of a polynomial will be a measure of its complexity. Another notion of complexity which may at first seem more natural is the quadratic norm. If $P(X_1, \ldots, X_n) = \sum a_{j_1 \cdots j_n} X_1^{j_1} \cdots X_n^{j_n}$ is a polynomial with complex coefficients, then the *quadratic norm* of P is

$$\|P\| = \sqrt{\sum |a_{j_1 \cdots j_n}|^2}.$$

The following lemma from [11] relates these two notions. Lemma 2.1.7 of [11] is as follows:

Lemma 5. *If $P \in \mathbb{C}[X_1, \ldots, X_n]$, then $M(P) \leq \|P\|$.*

Lemma 2.1.9 of [11] relates the Mahler measure and degree of $P \in \mathbb{C}[X]$ to the size of the coefficients of P.

Lemma 6. *Let $P(X) = c_0 + c_1 X + \cdots c_m X^m \in \mathbb{C}[X]$ be a polynomial in one variable. Then*

$$|c_i| \leq \binom{m}{i} M(P).$$

In particular $|c_0|, |c_m| \leq M(P)$.

Let $\alpha \in \mathbb{C}$ be an algebraic number of degree m and let $P(X)$ be its minimal polynomial over $\mathbb{Z}$. Define the measure $M(\alpha)$ of α to be

$$M(\alpha) = M(P).$$

Closely related to the measure of α is its *absolute multiplicative height*, $H(\alpha)$, given by the equation

$$H(\alpha) = M(\alpha)^{1/m}.$$

At times it is more convenient to consider the *absolute logarithmic height*, $h(\alpha)$, of an algebraic number α given by

$$h(\alpha) = \log H(\alpha) = \frac{1}{m} \log M(\alpha). \tag{5}$$

Let α and β be algebraic numbers. We have the following facts found in [17, Lemma 2A]

$$H(\alpha\beta) \leq H(\alpha)H(\beta).$$

$$H(\alpha + \beta) \leq 2H(\alpha)H(\beta).$$

Equivalently,

$$h(\alpha\beta) \leq h(\alpha) + h(\beta). \tag{6}$$

$$h(\alpha + \beta) \leq \log 2 + h(\alpha) + h(\beta). \tag{7}$$

There is a natural notion of height for vectors of algebraic numbers which is defined in [17, page 192]. For our purposes it will be enough to know that if $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_k)$ is a vector of algebraic numbers then for all i, $1 \leq i \leq k$

$$H(\alpha_i) \leq H(\boldsymbol{\alpha}),$$

and

$$h(\alpha_i) \leq h(\boldsymbol{\alpha}).$$

A highly nontrivial result due to Shou-Wu Zhang [20] is as follows:

Lemma 7. *Let $P_1, P_2, \cdots, P_n$ be polynomials in the variables $X_1, X_2, \cdots, X_k$. If $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \cdots, \alpha_k)$ is an isolated solution to the equations $P_i = 0$ then its absolute logarithmic height is bounded as follows:*

$$h(\boldsymbol{\alpha}) \leq A(n) \left(\sum_{i=1}^{n} \deg P_i \right) \left(\left(\sum_{i=1}^{n} \frac{M(P_i)}{\deg P_i} \right) + B(n) \log 2 \right)$$

where $A(n) = (n^2 - n + 1)! n^{-2} [(n-1)!]^{-n}$ and $B(n) = (n^3 - n^2)(n^2 - n + 1)^{-1}$.

Finally, a technical result that will be used in the discussion of the hyperbolic pieces is as follows:

Lemma 8. *Suppose $A = B_1 B_2 \cdots B_k$ is the product of the k matrices,*

$$B_i = \begin{pmatrix} \beta_{11}^i & \beta_{12}^i \\ \beta_{21}^i & \beta_{22}^i \end{pmatrix},$$

with each β_{jl}^i an algebraic number. Suppose further that $h(\beta_{jl}^i) \leq h$ for all i, j, l. Then for each entry, a_{ij}, of A

$$h(\alpha_{ij}) \leq (2^{k-1} - 1)(\log 2) + (2^k + 2^{k-1} - 2)h.$$

Proof. The proof is a straight-forward induction on k. Clearly the lemma holds for the case $k = 1$. Now suppose it is true for $A' = B_1 B_2 \cdots B_{k-1}$. Let $A = B_1 B_2 \cdots B_{k-1} B_k = A' B_k$. Notice that each entry of A is of the form $\alpha'_{i1} \beta_{1j}^k + \alpha'_{i2} \beta_{2j}^k$ where α'_{ij} is the i, jth entry of A' and β_{ij}^k is the i, jth entry of B_k. Using inequalities (6) and (7) we may conclude that

$$\begin{aligned} h(\alpha_{ij}) &= h(\alpha'_{i1} \beta_{1j}^k + \alpha'_{i2} \beta_{2j}^k) \\ &\leq \log 2 + h(\alpha'_{i1} \beta_{1j}^k) + h(\alpha'_{i2} \beta_{2j}^k) \\ &\leq \log 2 + h(\alpha'_{i1}) + h(\beta_{1j}^k) + h(\alpha'_{i2}) + h(\beta_{2j}^k) \\ &\leq \log 2 + 2h + 2(2^{k-2} - 1)(\log 2) + 2(2^{k-1} + 2^{k-2} - 2)h \\ &= (2^{k-1} - 1)(\log 2) + (2^k + 2^{k-1} - 2)h. \end{aligned}$$

□

5.2 Covering hyperbolic pieces

We are now ready to produce the desired covers of the hyperbolic pieces in the JSJ decomposition of our knot complement which do not satisfy Lemma 3. Let $D(n)$ be as in (3).

Theorem 2. *Suppose $N - \partial N$ has a complete, finite volume hyperbolic structure, and N is either the complement of a knot in S^3 or the complement of a*

knot in a solid torus with winding number 0*. If* N *has a combinatorial ideal triangulation with* n *tetrahedra then there is a number* $B \in \mathbb{N}$ *with* $B \leq D(n)$ *such that for every prime* p *not dividing* B*,* N *has a finite cover* $\widetilde{N}$ *with at most* $p^{3\left(2^{4n+4}(8n^2+4n)^{(4n+4)2^{4n+4}}\right)}$ *sheets in which each boundary component of* $\widetilde{N}$ *is the noncyclic* p^2*-sheeted cover of a boundary component of* N*.*

Proof. Suppose $N - \partial N$ has a complete, finite volume hyperbolic structure, and is either a piece of the JSJ decomposition of the complement of a knot in S^3 or a knot in a solid torus with winding number 0. Suppose further that N has a combinatorial ideal triangulation with n tetrahedra. Let p be a prime integer.

N has one or two boundary components: $\partial_0 N$ and possibly $\partial_1 N$. Ignoring questions of base points for the moment, we will produce a homomorphism from $\pi_1(N)$ to a finite group whose kernel will intersect each $\pi_1(\partial_k N)$ in the subgroup $p \cdot \pi_1(\partial_k N)$. The covering space of N corresponding to this kernel will be the desired cover.

More explicitly, let m_k and l_k be a meridian and longitude for $\partial_k N$. Choose the base point of $\partial_k N$ to be the point of intersection of m_k and l_k, and fix a path from the base point of N to the base point of $\partial_k N$. We get explicit inclusions $i_k^* : \pi_1(\partial_k N) \to \pi_1(N)$. Let $\lambda_k, \mu_k \in \pi_1(N)$ be the classes of m_k and l_k respectively. For an oriented loop, b, in the space U let $[b]_U \in H_1(U)$ denote its homology class. If no space U is indicated then we will assume the space is N. As in section 4 set the homomorphism

$$\bar{\theta} : \pi_1(N) \to H_1(N)/pH_1(N)$$

to be the composition of the Hurewicz map and the quotient map.

The manifold, N, is either the complement of a knot in S^3 or the complement of a knot with winding number 0 in the solid torus. In either case we have that $\bar{\theta}(\lambda_k) = 0$ and $\bar{\theta}(\mu_k)$ has order exactly p. Using the hyperbolic structure of N we will produce another homomorphism

$$\bar{\rho} : \pi_1(N) \to SL(2, F)$$

for F some finite field of characteristic p. By construction it will be clear that $\bar{\rho}(\lambda_k)$ has order exactly p, and $\bar{\rho}(\mu_k)$ has order dividing p. It follows directly that $\bar{\rho} \times \bar{\theta}$, will be a homomorphism with the desired kernel. The challenge will be to show that for all $p > D(n)$ such a $\bar{\rho}$ exists and to bound the minimum degree of the finite field F over $\mathbb{Z}/p\mathbb{Z}$ from above. Clearly, this will bound the order of the group $SL(2, F) \times \mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}$ which will contain the image of $\bar{\rho} \times \bar{\theta}$.

Produce a presentation of $\pi_1(N)$ as follows: Let C be the standard spine of N dual to $\mathcal{T}$. Clearly, C will have n vertices (one for each ideal tetrahedron of $\mathcal{T}$) and $4n/2 = 2n$ edges (one for each face of $\mathcal{T}$). Note that C is homotopy equivalent to N, and ∂N is a union of tori. It follows that C has Euler characteristic 0. This implies that C must have n faces. If we fix a maximal tree in the 1-skeleton of C, we get a presentation $\langle g_0, g_1, \ldots, g_n | r_1, r_2, \ldots, r_n \rangle$

for $\pi_1(C) \cong \pi_1(N)$ with $n+1$ generators and n relations. Furthermore, each edge of C is incident with 3 faces so the sum of the lengths of the relations must be $3(n+1)$.

The ideal triangulation, $\mathcal{T}$, of N with n ideal tetrahedra induces a natural triangulation of ∂N as follows: Place a single normal triangle in each corner of each ideal tetrahedron of $\mathcal{T}$. Gluing these triangles to form a normal surface gives a triangulation for ∂N with exactly $4n$ triangles. Dual to this triangulation is a polygonal decomposition of the boundary tori with $4n$ vertices whose 1-skeleton is a trivalent graph. For each boundary component, $\partial_k N$, we have paths x'_k, and y'_k in this 1-skeleton generating the fundamental group of that boundary component. In fact we may assume that these paths all have length at most $4n$ by insisting that x'_k and y'_k traverse each vertex at most once. These paths project in a natural way onto the 1-skeleton of the standard spine dual to $\mathcal{T}$. Whence we get words x_k and y_k in $g_0, g_1, \ldots, g_n$ generating the fundamental group of $\partial_k N$ as a subgroup of $\pi_1(N)$. Furthermore, the words x_k and y_k have length at most $4n$.

In order control the image of λ_k under $\bar{\rho}$, we will bound its length as a word in x_k and y_k. (We may assume after adjusting paths connecting base points that $\lambda_k \in \langle x_k, y_k \rangle$.)

Assume for the moment that N is a knot complement. Then the homology class $[l_0]$ is trivial. Consider the presentation $\langle g_0, g_1, \ldots, g_n | r_1, r_2, \ldots, r_n \rangle$ for $\pi_1(N)$. There is a corresponding abelian presentation for $H_1(N)$. For $0 \leq j \leq n$ choose $\nu_j \in \mathbb{Z}$ so that $[g_j] = \nu_j [m_0]$. Each relation, r_j, is trivial in $\pi_1(N)$ and hence must map to 0 in $H_1(N)$. This translates to an equation specifying that some integral linear combination of ν_j's is 0. For example, the relation $g_0 g_2 g_3 g_1^{-1} g_2 = 1$ would give the equation $1\nu_0 - 1\nu_1 + 2\nu_2 + 1\nu_3 = 0$. Consider the $n \times (n+1)$ matrix, B, whose ijth entry is the coefficient of ν_j in the equation coming from the relation r_i. The vector $(\nu_0, \cdots, \nu_n)$ will be the smallest nonzero, integral vector whose dot product with each row of B is 0. Since B is a presentation matrix for the homology of N, this property actually characterizes $(\nu_0, \cdots, \nu_n)$ up to sign. Let B_j be the $n \times n$ minor of B formed by dropping the jth column of B. I claim that the integer vector $(\det B_0, -\det B_1, \det B_2, \cdots, (-1)^n \det B_n)$ is a multiple of $(\nu_0, \cdots, \nu_n)$. Let $\mathbf{w} = (w_0, \cdots, w_n)$ be an arbitrary vector. By definition, $\mathbf{w}$ will be in the row space of B if and only if the determinant of the $n \times n$ matrix formed by adding $\mathbf{w}$ in as the first row of B is 0. Hence, $\mathbf{w}$ will be in the row space of B if and only if $\sum_{j=0}^{n} w_j (-1)^j \det(B_j) = 0$. This shows that $(\det B_0, -\det B_1, \det B_2, \cdots, (-1)^n \det B_n)$ is indeed perpendicular to the row space of B and must be a multiple of $(\nu_0, \cdots, \nu_n)$.

Recall that in a standard spine, an edge in the 1-skeleton is incident with exactly 3 faces. This implies that the sum of the absolute values of entries in a column of B is 3. It follows that $|\det(B_j)| \leq 3^n$ for each minor B_j. We have also shown that $|\nu_j| \leq |\det(B_j)|$ for all j, thus $|\nu_j| \leq 3^n$ for all j.

The words x_0 and y_0 are of length at most $4n$ in the g_i's. Hence if $a_0, b_0 \in \mathbb{Z}$ are chosen so that $[x_0] = a_0[m_0]$ and $[y_0] = b_0[m_0]$, then $|a_0|, |b_0| \leq 4n3^n$. In $H_1(\partial_0 N)$ we can write the homology class of l_0 as a linear combination of

$[x_0]_{\partial_0 N}$ and $[y_0]_{\partial_0 N}$ by noting that if $[l_0]_{\partial_0 N} = v[x_0]_{\partial_0 N} + w[y_0]_{\partial_0 N}$ then (v, w) generates the null space of the 1×2 matrix $(a_0\ b_0)$. In fact

$$\pm[l_0]_{\partial_0 N} = -b_0[x_0]_{\partial_0 N} + a_0[y_0]_{\partial_0 N} \tag{8}$$

where $|a_0|, |b_0| \leq 4n3^n$.

Now suppose N is the complement of a knot with winding number 0 in the solid torus. Then $H_1(N)$ has abelian presentation

$$H_1(N) = \left\langle [m_0], [m_1], [l_0], [l_1] \middle| [l_1] = 0, [l_0] = 0 \right\rangle$$

As above, let B be the presentation matrix for $H_1(N)$ coming from the presentation of $\pi_1(N)$. For each $k \in \{0, 1\}$ at least one of $[x_k]$ or $[y_k]$ must be nontrivial. Without loss of generality assume $[x_k]$ is nontrivial. Let $B^{(k)}$ be the $(n+1) \times (n+1)$ matrix whose first n rows agree with B and whose $(n+1)$th row comes from the word x_k. Let $\nu_0^{(k)}, \cdots, \nu_n^{(k)} \in \mathbb{Z}$ be integers such that

$$[g_i] = \nu_i^{(0)} \cdot [m_0] + \nu_i^{(1)} \cdot [m_1]$$

Then $(\nu_0^{(k)}, \cdots, \nu_n^{(k)})$ generates the null space of $B^{(1-k)}$. One of the first n rows of B is a linear combination of the others (over $\mathbb{Q}$). If we remove this row, $(\nu_0^{(k)}, \cdots, \nu_n^{(k)})$ still generates the null space of the new matrix. The same argument as above shows that $|\nu_j^{(k)}| \leq 16n^2 3^{n-1}$. We may then conclude that there are integers $a_k, b_k \in \mathbb{Z}$ such that

$$\pm[l_k]_{\partial_k N} = -b_k[x_k]_{\partial_k N} + a_k[y_k]_{\partial_k N} \tag{9}$$

where

$$|a_k|, |b_k| \leq 16n^2 3^{n-1}.$$

The hyperbolic structure of N gives a faithful representation of $\pi_1(N)$ in $PSL_2(\mathbb{C})$. As Thurston has shown, we may lift this representation to a faithful representation $\rho : \pi_1(N) \to SL_2(\mathbb{C})$. Let

$$\rho(g_k) = \begin{pmatrix} z_{4k} & z_{4k+1} \\ z_{4k+2} & z_{4k+3} \end{pmatrix}$$

Let $A = \mathbb{Z}[z_0, z_1, \cdots, z_{4n+3}] \in \mathbb{C}$. Since

$$\rho(g_k^{-1}) = \begin{pmatrix} z_{4k+3} & -z_{4k+2} \\ -z_{4k+1} & z_{4k} \end{pmatrix},$$

we actually have $\rho : \pi_1(N) \to SL_2(A)$. If we produce a ring homomorphism $\eta : A \to F$ for some finite field F then it will induce a group homomorphism $\eta* : SL_2(A) \to SL_2(F)$. Composing $\eta*$ with ρ will give $\bar{\rho} : \pi_1(N) \to SL(2, F)$. Our goal is to bound the height and degree of $\mathbf{z} = (z_0, z_1, \cdots, z_{4n+3}) \in \mathbb{C}^{4n+4}$, and extract a sufficient criterion on F for it to have a suitable nontrivial ring homomorphism $\eta : A \to F$.

The point $\mathbf{z} \in \mathbb{C}^{4n+4}$ satisfies the $n+1$ polynomial equations

$$G_k = X_{4k}X_{4k+3} - X_{4k+2}X_{4k+1} - 1 = 0$$

specifying that $\det \rho(g_k) = 1$.

Each relation r_k gives four polynomial relations satisfied by $\mathbf{z}$ indicating that $\rho(r_k)$ is the identity matrix. One of these equations is superfluous, so we drop it and let R_{3k}, R_{3k+1}, and R_{3k+2} be the remaining three. If the relation ρ_k has length l then the polynomials R_{3k},R_{3k+1}, and R_{3k+2} will have degree at most l and will be the sum of at most 2^l monomials with coefficient ± 1 and possibly the term -1.

After an appropriate conjugation we may assume without loss of generality that

$$\rho(x_0) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

Hence we get four more polynomial relations satisfied by $\mathbf{z}$. Again, we drop one of them and let C_0, C_1, and C_2 be the three that remain. By the same argument as above each polynomial relation C_i is a sum of at most 2^{4n} terms of degree at most $4n$ and possibly the term -1.

If N has two boundary components then by the completeness of N we must have $\mathrm{tr}(\rho(x_1)) = \pm 2$. We specify this with the single polynomial relation Q. The same argument gives that Q is a sum of at most 2^{4n} terms of degree at most $4n$ and the term ∓ 2.

In fact, $\mathbf{z} \in \mathbb{C}^{4n+4}$ is an isolated root of the $4n+4$ or $4n+5$ polynomials, $\mathcal{P} = \{G_0, \ldots, G_n,\ R_3, \ldots, R_{3n+2},\ C_0, C_1, C_2, Q\}$. (See [3, Proposition 2].) In light of Lemma 7, bounds on the Mahler measures of these polynomials will give a bound on the height of $\mathbf{z}$.

We now bound the Mahler measures of these polynomials. Firstly by Lemma 5,

$$M(G_k) = M(XY - ZW - 1) \leq \|XY - ZW - 1\| = \sqrt{3}$$

If the relation r_k has length l then polynomials R_{3k}, R_{3k+1}, and R_{3k+2} will be sums of 2^l monomials with coefficient ± 1 and possibly the term -1. Thus if $\mathbf{w}$ is a vector of complex numbers with norm 1 then clearly $|R_i(\mathbf{w})| < 2^l + 1$. We get the following bound.

$$\begin{aligned} M(R_i) &= \exp\left(\int_0^1 \cdots \int_0^1 \log|R_i(e^{2\pi i t_1}, \cdots, e^{2\pi i t_{4n+4}})|\, dt_1 \cdots dt_{4n+4}\right) \\ &\leq \exp\left(\int_0^1 \cdots \int_0^1 \log(2^l + 1)\, dt_1 \cdots dt_{4n+4}\right) \\ &= 2^l + 1. \end{aligned}$$

Similarly $M(C_i) \leq 2^{4n} + 1$, and $M(Q) \leq 2^{4n} + 2$.

We now have all the necessary ingredients to bound $h(\mathbf{z})$. The degree of each G_i is 2. The sum of the lengths of the relations r_i is $3(n+1)$, so the sum of the degrees of the R_i's is at most $9(n+1)$. Each C_i has degree at most $4n$, and Q has degree at most $4n$ as well. By Lemma 7,

$$
\begin{aligned}
h(z_k) &\le h(\mathbf{z}) \\
&\le A(4n+5)\left(\sum_{P\in\mathcal{P}} \deg P\right)\left(\left(\sum_{P\in\mathcal{P}} \frac{M(P)}{\deg P}\right) + B(4n+5)\log 2\right) \\
&\le A(4n+5)\left(2(n+1) + 3\cdot 3(n+1) + 3\cdot 4n + 4n\right) \qquad (10) \\
&\quad \cdot \left(\left(\tfrac{\sqrt{3}}{2}(n+1) + 3\cdot\left(2^{3(n+1)} + n\right) + 4\cdot 2^{4n}\right) + B(4n+5)\log 2\right) \\
&\le A(4n+5)(27n+5) \\
&\quad \cdot \left(2^{4n+2} + 3\cdot 2^{3n+3} + \left(\tfrac{\sqrt{3}}{2}+3\right)n + B(4n+5)\log 2 + \tfrac{\sqrt{3}}{2}\right).
\end{aligned}
$$

The ring homomorphism, $\eta : A \to F$, is given as follows. Let $W_k(X) \in \mathbb{Z}[X]$ be the minimal polynomial of z_k. Fix a prime, p, and let $W_k^{(p)}(X)$ denote the image of $W_k(X)$ under the natural map $\mathbb{Z}[X] \to \mathbb{F}_p[X]$ where $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$. If p does not divide the leading coefficient of $W_1(X)$ then we are assured that $W_1^{(p)}(X)$ has a root, ζ_1, in the algebraic closure of $\mathbb{F}_p$, and we have a ring homomorphism $\mathbb{Z}[z_1] \to \mathbb{F}_p(\zeta_1)$ taking z_1 to ζ_1. Set $A_1 = \mathbb{Z}[z_1]$ and $F_1 = \mathbb{F}_p(\zeta_1)$ In this case we get a ring homomorphism, $\eta_1 : A_1 \to F_1$. Let $\tilde{\eta}_1 : A_1[X] \to F_1[X]$ be the induced map on the polynomial rings.

Inductively let $S_{i+1}(X) \in A_i[X]$ be the minimal polynomial of z_{i+1} over A_i. If p does not divide the leading coefficient of $W_{i+1}(X)$ then $W_{i+1}^{(p)}(X)$ has roots in the algebraic closure of $\mathbb{F}_p$ some of which will be roots of $\tilde{\eta}_i\left(S_{i+1}(X)\right)$. Let ζ_{i+1} be one such root. Set $A_{i+1} = A_i[z_{i+1}]$ and $F_{i+1} = F_i(\zeta_{i+1})$. In this case we get a ring homomorphism $\eta_{i+1} : A_{i+1} \to F_{i+1}$ restricting to η_i on A_i and taking z_{i+1} to ζ_{i+1}. Let $\tilde{\eta}_{i+1} : A_{i+1}[X] \to F_{i+1}[X]$ be the induced map on the polynomial rings.

From this discussion it is clear that we will have a homomorphism, $\eta_{4n+4} : A_{4n+4} \to F_{4n+4}$, if p does not divide any of the leading coefficients of the W_k's. Of course, $A = A_{4n+4}$. Set $\eta = \eta_{4n+4}$ and $F = F_{4n+4}$. After further restriction on p, η will be the desired homomorphism.

We will now proceed to bound the degree $[F : \mathbb{F}_p]$. In [4] a bound on the degree of a polynomial in a Gröbner basis with any monomial order is given. This gives a bound on the degrees of the polynomials in a Gröbner basis of the ideal generated by $\mathcal{P}$ which in turn gives a bound on the degree of W_k.

$$
\deg(W_k) \le 2(8n^2+4n)^{2^{4n+4}}
$$

Thus,

$$
\begin{aligned}
[F : \mathbb{F}_p] &= \prod_{i=1} \deg\left[\tilde{\eta}_i\left(S_{i+1}(X)\right)\right] \\
&\le \prod_{i=1} \deg W_i \\
&\le 2^{4n+4}\left(8n^2+4n\right)^{(4n+4)2^{4n+4}}
\end{aligned}
$$

We also get a bound of the order of the field F.

$$|F| \leq p^{\left(2^{4n+4}\left(8n^2+4n\right)^{(4n+4)2^{4n+4}}\right)}.$$

The order of $SL_2(F)$ is $(|F|^2-1)(|F|-1)$; hence,

$$|SL_2(F)| \leq |F|^3 \leq p^{3\left(2^{4n+4}\left(8n^2+4n\right)^{(4n+4)2^{4n+4}}\right)}. \tag{11}$$

We now address the question of how to ensure that p meets all of the conditions stipulated above. Inequalities (10) and (11) combine to bound the Mahler measure of z_k.

$$\begin{aligned} M(z_k) \quad \leq \quad & \exp\Big[2\left(8n^2+4n\right)^{2^{4n+4}} A(4n+5)(27n+5) \\ & \cdot \left(2^{4n+2}+3\cdot 2^{3n+3}+\left(\tfrac{\sqrt{3}}{2}+3\right)n+B(4n+5)\log 2+\tfrac{\sqrt{3}}{2}\right)\Big]. \end{aligned} \tag{12}$$

If p does not divide the coefficient the highest degree term of W_k then S_k has roots in Ω_p. Lemma 6 shows that the coefficient of the highest degree term of W_k is less than $M(W_k)$. This is sufficient to ensure that we have a ring homomorphism $\eta : A \to F$. However, we need further restrictions on p to ensure that $\rho(l_i)$ is nontrivial.

Recall that $[l_0]_{\partial_0 N} = -b_0[x_0]_{\partial_0 N} + a_0[y_0]_{\partial_0 N}$ for some $a_0, b_0 \in \mathbb{Z}$ with $|a_0|, |b_0| \leq 16n^2 3^{n-1}$.

$$\rho(x_0) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

and

$$\rho(y_0) = \begin{pmatrix} 1 & \alpha_0 \\ 0 & 1 \end{pmatrix}$$

The word y_0 has length at most $4n$ so by Lemma 8

$$\begin{aligned} h(\alpha_0) \quad \leq \quad & (2^{4n-1}-1)(\log 2) + (2^{4n}+2^{4n-1}-2)A(4n+5)(27n+5) \\ & \cdot \left(2^{4n+2}+3\cdot 2^{3n+3}+\left(\tfrac{\sqrt{3}}{2}+3\right)n+B(4n+5)\log 2+\tfrac{\sqrt{3}}{2}\right). \end{aligned}$$

It follows that

$$\rho(l_0) = \begin{pmatrix} 1 & -b_0+a_0\alpha_0 \\ 0 & 1 \end{pmatrix}$$

and

$$\begin{aligned} h(-b_0+a_0\alpha_0) \quad \leq \quad & \log 2 + \log a_0 + \log b_0 + h(\alpha) \\ \leq \quad & \log 2 + 2\log(16n^2 3^{n-1}) + (2^{4n-1}-1)(\log 2) \\ & +(2^{4n}+2^{4n-1}-2)A(4n+5)(27n+5) \\ & \cdot \left(2^{4n+2}+3\cdot 2^{3n+3}+\left(\tfrac{\sqrt{3}}{2}+3\right)n+B(4n+5)\log 2+\tfrac{\sqrt{3}}{2}\right) \end{aligned}$$

The degree of $-b_0 + a_0\alpha_0$ is at most the product of the degrees of the z_k's; hence,

$$\deg(-b_0+a_0\alpha_0) \leq 2^{4n+4}\left(8n^2+4n\right)^{(4n+4)2^{4n+4}}.$$

From (5) we get:

$$
\begin{aligned}
&M(-b_0+a_0\alpha_0) \leq \\
&\quad \exp\Big[2^{4n+4}\left(8n^2+4n\right)^{(4n+4)2^{4n+4}}\Big(\log 2+2\log(16n^2 3^{n-1}) \\
&\quad +(2^{4n-1}-1)(\log 2)+(2^{4n}+2^{4n-1}-2)A(4n+5)(27n+5) \\
&\quad \cdot\left(2^{4n+2}+3\cdot 2^{3n+3}+(\tfrac{\sqrt{3}}{2}+3)\,n+B(4n+5)\log 2+\tfrac{\sqrt{3}}{2}\right)\Big)\Big].
\end{aligned} \tag{13}
$$

If p does not divide the constant term of the minimal polynomial of $-b_0 + a_0\alpha_0$ over $\mathbb{Z}$ then $\eta(-b_0 + a_0\alpha_0)$ cannot be 0. The order of $\bar{\rho}(l_0)$ is the additive order of $\eta(-b_0 + a_0\alpha_0)$ which must be p.

In the case that N has two boundary components, we also require $\bar{\rho}(l_1)$ to have order exactly p. Here we note that $\rho(x_1)$ is parabolic and so

$$
\rho(x_1) = \begin{pmatrix} a & -b \\ -c & d \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a & -b \\ -c & d \end{pmatrix}^{-1} = \begin{pmatrix} 1+ac & a^2 \\ -c^2 & 1-ac \end{pmatrix}
$$

for some $a, c \in \mathbb{C}$. Adjoin square roots of the upper right and lower left entries of $\rho(x_1)$ to A get the ring A', and extend $\eta : A \to F$ to some ring homomorphism $\eta' : A' \to F'$ (no further restriction on p is needed for such an η' to exist.) Then in $SL_2(F')$, $\bar{\rho}(x_1)$ will be conjugate to $\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ as long as at least one of $\eta(a^2)$ or $\eta(c^2)$ is nonzero. At least one of a^2 and c^2 is nonzero in $\mathbb{C}$. Assume, without loss of generality, that $a^2 \neq 0$. The word x_1 has length at most $4n$. As above this gives a bound on the height and degree of a^2 which gives the following bound on the Mahler measure of a^2

$$
\begin{aligned}
&M(a^2) \leq \\
&\quad \exp\Big[2^{4n+4}\left(8n^2+4n\right)^{(4n+4)2^{4n+4}}\Big((2^{4n-1}-1)(\log 2) \\
&\quad +(2^{4n}+2^{4n-1}-2)A(4n+5)(27n+5) \\
&\quad \cdot\left(2^{4n+2}+3\cdot 2^{3n+3}+(\tfrac{\sqrt{3}}{2}+3)\,n+B(4n+5)\log 2+\tfrac{\sqrt{3}}{2}\right)\Big)\Big].
\end{aligned} \tag{14}
$$

If p does not divide the constant term of the minimal polynomial of a^2 over $\mathbb{Z}$ then $\eta(a^2)$ cannot be 0. The above bound on the Mahler measure of a^2 is also a bound on the constant term in the minimal polynomial of a^2.

Now consider $\rho(y_1)$. If a, b, c, d are as above, we must have $\alpha_1 \in \mathbb{C}$ such that

$$
\rho(y_1) = \begin{pmatrix} a & -b \\ -c & d \end{pmatrix}\begin{pmatrix} 1 & \alpha_1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a & -b \\ -c & d \end{pmatrix}^{-1}
$$

If we reverse the labels on the boundary components for a moment it is clear that the height and degree of α_1 satisfy the same bounds as the ones given for height and degree of α_0. Again we have $a_1, b_1 \in \mathbb{Z}$ with $|a_1|, |b_1| \leq 16n^2 s^{n-1}$ such that $[l_1]_{\partial_1 N} = -b_1[x_1]_{\partial_1 N} + a_1[y_1]_{\partial_1 N}$. If follows that

$$
\rho(l_1) = \begin{pmatrix} a & -b \\ -c & d \end{pmatrix}\begin{pmatrix} 1 & -b_1 + a_1\alpha_1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a & -b \\ -c & d \end{pmatrix}^{-1}
$$

$M(-b_1 + a_1\alpha_1)$ satisfies the same bound as was given for $M(-b_0 + a_0\alpha_0)$. If p does not divide the constant term of the minimal polynomial of $-b_1 + a_1\alpha_1$, then $\bar{\rho}(l_1)$ has order p.

In summary, if p does not divide the top degree terms of the minimal polynomials of the z_i's with $1 \le i \le 4n+4$ then $\eta : A \to F$ exists. If p does not divide the constant terms of minimal polynomials of $-b_0 + a_0\alpha_0$, a^2, and $-b_1 + a_1\alpha_1$ then $\bar{\rho}(l_i)$ has order exactly p. Let B be the product of all these coefficients. From Lemma 6 and inequalities (12), (13), and (14) it follows that

$$\begin{aligned} B \quad \le \quad & \exp\Big[2(4n+4)\left(8n^2+4n\right)^{2^{4n+4}} A(4n+5)(27n+5) \\ & \cdot\left(2^{4n+2} + 3\cdot 2^{3n+3} + \left(\tfrac{\sqrt{3}}{2}+3\right)n + B(4n+5)\log 2 + \tfrac{\sqrt{3}}{2}\right) \\ & + 2^{4n+4}\left(8n^2+4n\right)^{(4n+4)2^{4n+4}}\Big(2\log 2 + 4\log(16n^2 3^{n-1}) \\ & + 3(2^{4n-1}-1)(\log 2) + 3(2^{4n}+2^{4n-1}-2)A(4n+5)(27n+5) \\ & \cdot\left(2^{4n+2} + 3\cdot 2^{3n+3} + \left(\tfrac{\sqrt{3}}{2}+3\right)n + B(4n+5)\log 2 + \tfrac{\sqrt{3}}{2}\right)\Big)\Big]. \\ = \quad & D(n). \end{aligned}$$

If p does not divide B then $\bar{\rho} \times \bar{\theta}$ has the desired kernel. □

6 Seifert Fibered Pieces

6.1 Seifert fibered pieces of a knot complement

We now turn our attention to Seifert fibered pieces of our knot complement.

Lemma 9. *Suppose N is a piece in the JSJ decomposition of the complement of a nontrivial, decompositionally linear knot in S^3, and N is Seifert fibered. Then N has base orbifold either a disk with two cone points of order u and v with $(u, v) = 1$, or N has base orbifold an annulus with one cone point.*

Proof. As in the statement of the lemma we assume N is a Seifert fibered piece in the JSJ decomposition of the complement of a decompositionally linear knot in S^3. Then N has one or two boundary components.

Case 1. Assume N has one boundary component. Then since N is embedded in S^3, the boundary of N is a torus in S^3. By the Solid Torus Theorem (see [16, page 107]), $\partial N \subset S^3$ bounds a solid torus on at least one side. Clearly, N cannot be a solid torus (since we assumed the knot to be nontrivial), so $S^3 - N$ must be a solid torus. This shows that N is actually the complement of a knot, K', in S^3. The only knots with Seifert fibered complements are torus knots (See [9, Theorem 10.5.1]). Hence, for some relatively prime $u, v \in \mathbb{Z}$, the knot K' is the (u, v)-torus knot. It follows that N has a Seifert fibered structure with two singular fibers with orders u and v. Thus, the base orbifold of N is a disk with two cone points with relatively prime orders u and v.

Case 2. Assume N has two boundary components. The manifold, N is *simple* in the sense that it contains no essential tori (otherwise we would have

cut along them). As in [9, Proposition C.5.2], the only simple Seifert fibered manifolds with two boundary components have base orbifold an annulus with a single cone point. □

Note that in the case where N has two boundary components, it is the complement of a knot in the solid torus with winding number equal to the order of the cone point of its base space. This situation is covered by Lemma 3.

6.2 Seifert fibered pieces with one boundary component

Now one case remains. If our knot K is a satellite of the (u, v)-torus knot then the JSJ decomposition of its complement has a Seifert fibered piece whose base space is a disk with two cone points. We will bound u and v based on the crossing number of K. The bridge number of K must be greater than the bridge number of the (u, v)-torus knot. The bridge number of a torus knot is known to be the smaller of $|u|$ and $|v|$ (See [12, Theorem 7.5.3]); consequently, if c is the number of crossings in some diagram of K then the smaller of $|u|$ and $|v|$ must be less than c. Unfortunately, we must bound the larger of the two.

Recall that Lemma 2 gives a bound on the number of tetrahedra needed to triangulate any piece in the JSJ decomposition of the complement of the knot K. We will proceed to bound the minimum number of tetrahedra needed to triangulate the complement of a (u, v)-torus knot from below. This will be done by showing that a knot complement that can be triangulated with n ideal tetrahedra has an Alexander polynomial with degree at most $(n^2 + n)3^{n+1}$.

Lemma 10. *Suppose L is a knot in S^3, and its complement $N = S^3 - L$ can be triangulated with n ideal tetrahedra. Then the Alexander polynomial of L has degree at most $(n^2 + n)3^{n+1}$.*

Proof. Let L be a knot in S^3 and $N = S^3 - L$ its complement. Suppose N has an ideal triangulation $\mathcal{T}$ with n ideal tetrahedra. Set $G = \pi_1(N)$. As in the proof of Theorem 2, we have a presentation $\langle g_0, g_1, \dots, g_n | r_1, r_2, \dots, r_n \rangle$ for G with $n + 1$ generators and n relations. Furthermore, each edge of C is incident with 3 faces so the sum of the lengths of the relations must be $3(n + 1)$.

Following the technique given in [1, example 9.15] this presentation of the group may be used to find the first elementary ideal of the Alexander module of K. To begin we let $F = \langle g_0, g_1, \dots, g_n \rangle$ be the free group. Let the derivations $\frac{\partial}{\partial g_i} : \mathbb{Z}F \to \mathbb{Z}F$ be the linear maps satisfying the following rules for all $\alpha, \beta \in F$:

- $\frac{\partial}{\partial g_i}(g_j) = \delta_{ij}$.
- $\frac{\partial}{\partial g_i}(\alpha^{-1}) = \alpha^{-1} \frac{\partial}{\partial g_i}(\alpha)$.
- $\frac{\partial}{\partial g_i}(\alpha \cdot \beta) = \frac{\partial}{\partial g_i}(\alpha) + \alpha \frac{\partial}{\partial g_i}(\beta)$.

For each generator g_i and each relation r_j compute $\frac{\partial}{\partial g_i} r_j$. Let $\psi : \mathbb{Z}F \to \mathbb{Z}G$ be the linear extension of the quotient homomorphism, $F \to G$, and $\varphi : \mathbb{Z}G \to \mathbb{Z}\langle t \rangle$ be the linear extension of the Hurewicz homomorphism, $G \to H_1(N) =$

$\langle t\rangle$. By [1, Proposition 9.14] we know that the ideal of $\mathbb{Z}\langle t\rangle$ generated by the determinants of the $n\times n$ minors of $A = \left(\varphi\circ\psi(\frac{\partial r_j}{\partial g_i})\right)$ will be a principal ideal generated by the Alexander polynomial, $\Delta(t)$, of L. Let A_i be the $n\times n$ minor of A got by removing the ith column. $\Delta(t)$ divides $\det A_i$ so if $d = \max\{\deg(\det A_i)\}$ then $\deg\Delta(t)\le d$. If each entry of A has degree l or less then $\deg(\det A_i)\le nl$, so all that remains is to bound the degrees of the entries of A.

Let us consider an entry, $a_{ij} = \varphi\circ\psi(\frac{\partial r_j}{\partial g_i})$, of A. As noted above, r_j is a word in the g_k's of length at most $3(n+1)$. Applying the rules above it is clear that $\frac{\partial}{\partial g_i}r_j$ is a linear combination of words in the g_k's with lengths bounded by $3(n+1)$. The map $\varphi\circ\psi$ takes a word in the g_k's to t^a where $a\in\mathbb{Z}$ is the number of times the path represented by the word winds around the knot L. Choose the integers ν_k so that $\varphi\circ\psi(g_k) = t^{\nu_k}$. If $|\nu_k|\le\nu$ for all k then $\deg(a_{ij})\le\nu 3(n+1)$.

In the proof of Theorem 2 we saw that $|\nu_j|\le 3^n$. Hence,

$$\deg(a_{ij})\le 3^n\cdot 3(n+1),$$

and hence

$$\deg\Delta(t)\le 3^{n+1}n(n+1).$$

□

In Lemma 2 we saw that a piece in the JSJ decomposition of our manifold has at most $25\cdot 4c = 100c$ tetrahedra. The above lemma tells us that if the piece is a knot complement then its Alexander polynomial has width at most $((100c)^2+100c)3^{100c+1}$. The Alexander polynomial of a (u,v)-torus knot has degree $(u-1)(v-1)$ (See [1, example 9.15]). Clearly $u,v\ge 2$, whence, $(u-1)\le(u-1)(v-1)$ and $(v-1)\le(u-1)(v-1)$. From these inequalities we get that $uv = (u-1)(v-1)+(u-1)+(v-1)+1\le 3(u-1)(v-1)+1$. It follows that

$$uv\le((100c)^2+100c)3^{100c+2}+1 \tag{15}$$

6.3 Covering Seifert fibered pieces

Now that we have bounded the orders of the cone points of in the base orbifolds of our Seifert fibered pieces we may proceed to produce the desired covers of these pieces.

Lemma 11. *Let N be Seifert fibered with base orbifold a disk with two cone points of order u and v. Then for each prime $p>3$, N has a cover, $\widetilde{N}$, with at most $2uvp^2$ sheets in which each boundary component of $\widetilde{N}$ is the noncyclic cover with p^2 sheets of a boundary component of N.*

Proof. Let N satisfy the hypotheses of the lemma, and let F be the base orbifold of N. F is a disk with two cone points of order u and v.

Glue a disk with one cone point of order $p>3$ to F to get a sphere F' with 3 cone points with orders u, v, and p. The orbifold, F', is hyperbolic, and by

[5] has a finite orbifold cover which is a manifold. In fact, [5] gives such a cover, $\widetilde{F'}$, with at most $2 \cdot \mathrm{LCM}(u, v, p)$ sheets. By removing open disk neighborhoods of each of the points of $\widetilde{F'}$ mapping to the cone point of F' with order p we get a cover $\widetilde{F}$ of F. By construction each boundary component of $\widetilde{F}$ is the p-fold cover of the boundary component of F. This shows that N has a cover, N_0, with at most $2uvp$ sheets whose base orbifold is the manifold, $\widetilde{F}$, and whose S^1 fibers map homeomorphically to the regular S^1 fibers of N. We may then take $\widetilde{N}$ to be the p-fold cover of N_0 whose base space is again $\widetilde{F}$ and whose S^1 fibers are p-fold covers of the S^1 fibers of N_0. Clearly $\widetilde{N}$ is the desired cover and has at most $2uvp^2$ sheets. $\square$

Lemma 11 and Inequality (15) combine to give the following theorem:

Theorem 3. *Let N be a Seifert fibered piece in the JSJ decomposition of the complement of a nontrivial knot, K, with a diagram with c crossings. Let p be any prime greater than 3. Then N has a cover $\widetilde{N}$ with $((20000c^2+200c)3^{100c+2}+2)p^2$ sheets or less whose boundary components are the noncyclic covers of order p^2 of the boundary components of N.*

Notice that the bound given in Theorem 3 is exponential in the crossing number. It could be made polynomial if it were known that the crossing number of a satellite knot cannot be less than the crossing number of its companion. It is conjectured that this should be true, but it has remained unproven since Schubert introduced the notion of satellite knots (See [10, Problem 1.67]).

7 Assembling the Covering Space

Now that we have produced the desired covers for the geometric pieces of our knot complement, we must show that they can be assembled to produce a cover of the entire complement. This will be done exactly as in [7, section 2]. Note that this cover will in general not be regular.

Recall that K is a decompositionally linear knot in S^3, $M = S^3 - K$ its complement, and $\{T_i\}_{i=1}^r$ a set of tori cutting M into geometric pieces $M_0, M_1, \ldots, M_r$ (See the bottom of Figure 6). Fix a prime p. Each boundary component of M_i is a torus with fundamental group isomorphic to $\mathbb{Z} \times \mathbb{Z}$. This group has a characteristic subgroup P of index p^2 generated by $(p, 0)$ and $(0, p)$. For each i, let $\widetilde{T_i}$ be the cover of T_i associated to the subgroup of $\pi_1(T_i)$ corresponding to $P \trianglelefteq \mathbb{Z} \times \mathbb{Z}$. Suppose for each i we produce a cover, $\widetilde{M_i}$, of M_i such that the boundary components of $\widetilde{M_i}$ are all covers equivalent to $\widetilde{T_j}$ for some j (see Figure 6). Then by taking sufficiently many copies, $\widetilde{M_i^k}$, of $\widetilde{M_i}$ we may assemble a cover $\widetilde{M}$ of M (see Figure 7). In fact if m_i is the number of sheets in the cover $\widetilde{M_i}$ then we will need $\mathrm{LCM}(\frac{m_0}{p^2}, \frac{m_1}{p^2}, \ldots, \frac{m_r}{p^2})/(\frac{m_i}{p^2})$ copies of $\widetilde{M_i}$, and $\widetilde{M}$ will have $p^2 \cdot \mathrm{LCM}(\frac{m_0}{p^2}, \frac{m_1}{p^2}, \ldots, \frac{m_r}{p^2})$ sheets. It is immediate that $\widetilde{M}$ is not a cyclic cover of M since the boundary components of $\widetilde{M}$ are not cyclic covers the boundary of M.

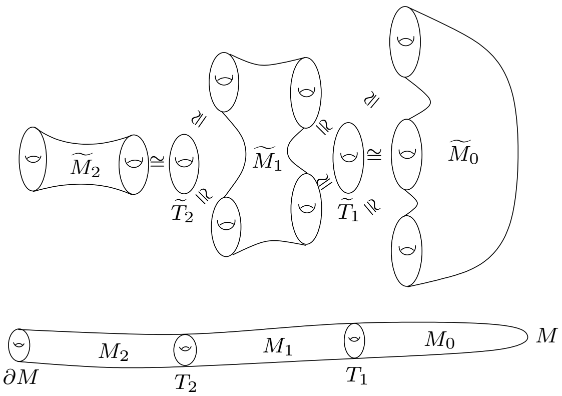

Figure 6: Pieces of the cover

We will now find a prime p for which we can produce such a set of coverings $\{\widetilde{M}_i\}_{i=0}^r$. Let us assume that our knot, K, has a diagram with c crossings and hence by Lemma 1 its complement, M, may be triangulated with $t \leq 4c$ tetrahedra. Corollary 1 showed that $r \leq 2t \leq 8c$ and that the M_i's can all be ideally triangulated with a total of $25t \leq 100c$ or less tetrahedra. If M_i satisfies the hypotheses of Lemma 3 then there is a number w_i with $w_i \leq \frac{c}{2}$ such that for every prime, p, not dividing w_i, Lemma 3 gives a cover, $\widetilde{M}_i$, with p^2 sheets. If M_i does not satisfy the hypotheses of Lemma 3 and is hyperbolic then Theorem 2 implies that there is a number $B_i \in \mathbb{N}$ with $B_i \leq D(100c)$ such that for every prime not dividing B_i there is a cover, $\widetilde{M}_i$, of M_i which has at most $p^{3\left(2^{4n+4}\left(8n^2+4n\right)^{(4n+4)2^{4n+4}}\right)}$ sheets all of whose boundary tori are the noncyclic p^2 cover of a boundary torus of M_i. If M_i is a torus knot complement then Theorem 3 gives such a cover $\widetilde{M}_i$ with at most $((20000c^2 + 200c)3^{100c+2} + 2)p^2$ sheets for all primes $p \geq 3$. We wish to find a prime p not dividing B_i for any $1 \leq i \leq r$. The function D is super exponential so if M_i has t_i tetrahedra and $\sum t_i \leq 100c$ then $\prod D(t_i)$ will be greatest when all tetrahedra are in a single M_i. Hence,

$$\prod w_i \prod B_i \leq \left(\tfrac{c}{2}\right)^{8c} D(100c).$$

The product of all the primes less than $x > 2$ is at least $e^{x/87}$ (See [8, page 85]) so there must be a prime p less than $87\left(\log(D(100c)) + 8c\log\frac{c}{2}\right)$ which does not divide any B_i or w_i.

Each geometric piece M_i has a cover $\widetilde{M_i}$ whose boundary components are the noncyclic p^2-sheeted covers of the boundary components of M_i.

From copies of these pieces we may assemble a cover $\widetilde{M}$ of M with at most

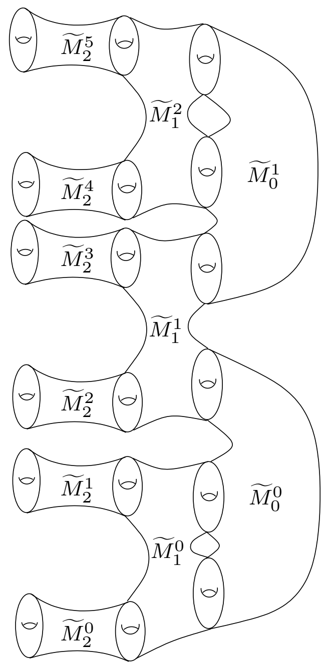

Figure 7: An assembled cover, $\widetilde{M}$, of M

$p^2 \cdot \mathrm{LCM}(\frac{m_0}{p^2}, \frac{m_1}{p^2}, \ldots, \frac{m_r}{p^2})$ sheets.

$$\begin{aligned} & p^2 \cdot \mathrm{LCM}\left(\frac{m_0}{p^2}, \frac{m_1}{p^2}, \ldots, \frac{m_r}{p^2}\right) \\ \leq \quad & m_0 m_1 \cdots m_r / p^{2r-2} \\ \leq \quad & \left(87\left(\log(D(100c)) + 8c\log\tfrac{c}{2}\right)\right)^{24c\left(2^{4n+4}\left(8n^2+4n\right)^{(4n+4)2^{4n+4}}\right)}. \end{aligned} \tag{16}$$

Thus we have completed the proof of Theorem 1.

8 Conclusions

We have shown that every decompositionally linear c-crossing knot has a finite non-cyclic cover with at most $\Phi(c)$ sheets. It should be noted that this is very much a worst case result. For example, the Alexander polynomial of a knot determines when the fundamental group of its complement surjects onto a dihedral group (See [1] 14.8).

This result raises a number of questions. Firstly, it would be nice to have such a bound for all knots. The bound $\Phi(c)$ seems to be far from tight. It would be interesting to improve this bound. From the other direction one might try to

produce lower limits for such a bound. This could be addressed by producing an infinite class of examples with a large number of sheets in the smallest finite noncyclic cover relative to the minimal crossing number. From an algorithmic point of view one might ask how to find and verify noncyclic covers efficiently.

A Appendices

A.1 An ideal triangulation of a knot complement

For completeness we construct a triangulation of a knot complement from a diagram of the knot. This discussion is largely based on the ideal triangulation algorithm in Jeffery Week's program, SnapPea. However, we will present it from the viewpoint of standard spines.

Proof of Lemma 1. Let K be a knot in S^3 with a diagram with $c > 0$ crossings. We will produce a standard spine based on this projection which will have less than $4c$ singular vertices. This spine will be dual to an ideal triangulation of the knot complement with less than $4c$ ideal tetrahedra.

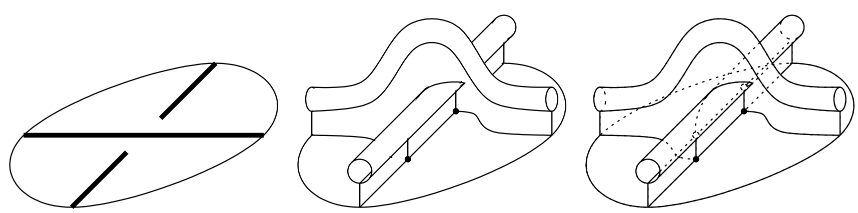

Figure 8: A knot projection and two views of the corresponding spine C'

We will assume that the projection of K is a 4-valent graph in the equatorial 2-sphere, S, in S^3 with crossing information at each vertex. Create a spine, C', as shown in Figure 8. Note that C' has exactly $4c$ singular vertices. For convenience we will assume that K is the core curve of the "tubes" in Figure 8. It should be clear that C' satisfies the neighborhood condition. As long as our projection contains at least one crossing, the complement in C' of its singular 1-skeleton will be a union of disks. However, C' is not a spine of $M = S^3 - K$ since two components of $M - C'$ are separated from ∂M.

We now modify C' to produce a standard spine of M. If one imagines C' to be a soap-bubble film the modifications we will make amount to "popping" two of the walls. Let C'_1 be the singular 1-skeleton of C'. Note that C' divides S^3 into 3 components. Two are homeomorphic to open 3-balls, and one is an open regular neighborhood of K which we will denote by N. Choose a disk, D_1, in $\partial N - C'_1$. The closure of D_1 in S^3 contains a maximal open arc, λ, such that $D_1 \cup \lambda$ is an open annulus. Let D_2 be the open disk in $C' - (C'_1 \cup D_1)$ whose

closure contains λ. Set $C'' = C' - (D_1 \cup \lambda \cup D_2)$, and let C_1'' be its singular 1-skeleton. Choose an open disk D_3 in $S - C_1''$. Let $C = C'' - D_3$.

The spine, C, is, in fact, a standard spine for M. To prove this we first show that M collapses to C. It should be apparent than $\bar{N} - K$ collapses to ∂N. The removal of $(D_1 \cup \lambda \cup D_2)$ from C' joins $N - K$ to a region homeomorphic to an open solid torus along an open annulus. The closure in M of this new region collapses to it's boundary. Finally, the removal of D_3 joins an open 3-ball to this region along an open disk. This joined region is $M - C$ and we see that its closure in M collapses to C.

Now we must show that C is a standard spine. The modification from C' to C preserves the neighborhood condition. Let C_1 be the singular 1-skeleton of C. We will see that $C'' - C_1''$ is a disjoint union of open disks and then that $C - C_1$ is as well.

First consider the effect of removing $(D_1 \cup \lambda \cup D_2)$ from C' to get C''. The equatorial 2-sphere, S, intersects C_1' in a graph which agrees with the actual knot projection except near crossings. In fact, C_1' and the knot projection cut S into nearly identical 2-disks. In S, the removal of $(D_1 \cup \lambda \cup D_2)$ eliminates the arc in C_1' corresponding to the projection of λ onto S. This has the effect of joining certain disks in $S - C_1'$. These joined regions will all be disks because the knot diagram will still be connected after the removal of any one over-arc. The other change after the removal of $(D_1 \cup \lambda \cup D_2)$ is that the vertical walls at both ends of λ will be joined to the pieces of the tunnel on the other side of the wall from λ. This amounts to gluing a disk to a disk along an arc in their boundaries. The results are still disks. Hence, $C'' - C_1''$ is a disjoint union of open disks.

Now consider the effect of removing D_3 from C'' to get C. Here each vertical wall surrounding D_3 will be joined to the disk in S on the other side of the wall from D_3. These disks are joined along single arcs in their boundary; therefore, they glue together to form disks. Consequently, $C - C_1$ is a union of open disks.

Let C_0 be the set of singular vertices of C. I claim that $C_1 - C_0$ must be a disjoint union of open arcs. If not then $C_1 - C_0$ contains an S^1. The spine, C, is connected and $C - C_1$ is composed entirely of disks so each disk must, in fact, have this S^1 as its boundary. This is impossible because the only such spine satisfying the neighborhood condition is composed of three disks glued along their boundary. This is not a spine of the complement of a knot in the 3-sphere or the complement of a knot in a solid torus. Consequently, C is a standard spine of $M - K$.

The standard spine, C, has strictly fewer singular vertices than C', so C has less than $4c$ singular vertices. It follows that there is a dual ideal triangulation of M with less than $4c$ ideal tetrahedra. $\square$

A.2 Homology calculations

Here we will calculate the first homology group of a piece in the JSJ decomposition of a knot complement. In section 3.3 we saw that M_0 will always be the complement of a knot in S^3. A well-known Mayer-Vietoris argument demon-

strates that $H_1(M_0) \cong \mathbb{Z}$. The following lemma gives the homology of pieces with two boundary components.

Lemma 12. *Suppose N is the complement of an open, regular neighborhood of a knot L in a solid torus. Suppose further that L has winding number w in the solid torus. Then $H_1(N)$ has abelian presentation*

$$\begin{aligned} H_1(N) &= \Big\langle [m_0],[m_1],[l_0],[l_1] \Big| [l_1] = w\cdot[m_0], [l_0] = w\cdot[m_1] \Big\rangle \\ &= \langle [m_0],[m_1] \rangle. \end{aligned}$$

Proof. As in the statement of the lemma, let N be the complement of an open, regular neighborhood of a knot L in a solid torus V. Suppose that L has winding number w in the solid torus. Fix an embedding of V in S^3. Let $\partial_0 N$ and $\partial_1 N$ be the boundary components of N as in section 3.3, and let m_k and l_k be a meridian and longitude in $\partial_k N$. For any curve α in N, let $[\alpha] \in H_1(N)$ be its homology class. Set X to be an open regular neighborhood of L in V which intersects $\mathring{N}$ in a regular neighborhood of $\partial_0 N$. Since $X \cup \mathring{N} = V$ we have the reduced homology Mayer-Vietoris sequence

$$0 \to H_1(\partial_0 N) \to H_1(N) \oplus H_1(X) \to H_1(V) \to 0.$$

This exact sequence shows that $H_1(N)$ is generated by $H_1(\partial_0 N)$ and $H_1(V)$. Of course $H_1(V)$ is generated by $H_1(\partial_1 N)$, so we conclude that $H_1(N)$ is generated by $H_1(\partial_0 N)$ and $H_1(\partial_0 N)$.

The curve l_1 bounds a disk, D, in V. One should observe that $D \cap N$ demonstrates that $[l_1] = w\cdot[m_0]$. Longitude l_0 is homologically unlinked with L in S^3. Let F be a Seifert surface for l_0 in S^3. The surface, $F \cap N$, demonstrates that $[l_0] = w\cdot[m_1]$. It follows that $\langle [m_0],[m_1]\rangle = H_1(N)$. One sees that $[m_1]$ has infinite order in $H_1(N)$ by noting that its image in $H_1(V)$ has infinite order. Similarly, the image of $[m_0]$ has infinite order in $H_1(S^3 - L)$ which implies that $[m_0]$ has infinite order in $H_1(N)$. Clearly, $[m_0]$ has trivial image in $H_1(V)$, and $[m_1]$ has trivial image in $H_1(S^3 - L)$. This shows that $[m_1]$ and $[m_0]$ are independent over the integers. ☐

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